Return Probability to the origin

GdT GGT and its coarse neighbourhood

1) Introduction

(M, g) compact Riemannian manifold.

$\mathbb{R}^n$ $(\tilde{M}, \tilde{g})$ universal covering

$\downarrow$ $\qquad \Gamma = \pi_1(M)$, finitely generated.

$\mathbb{R}^n/\mathbb{Z}^n$ (M, g)

Heat Kernel on $\tilde{M}$:

$f \in L^2(\tilde{M}, dvol_{\tilde{g}})$

$$e^{-t\Delta} f = \int_{\tilde{M}} h_t(x,y) f(y) \, dvol_{\tilde{g}}(y)$$

Heat semigroup

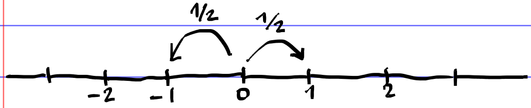
$h_t(x,y)$ heat kernel

asymptotic behaviour when $t \to \infty$ of $\sup_{x \in \tilde{M}} h_t(x,x)$

Saloff-Coste, Pittet, Coulhon, Varopoulos, Grigoryan

Is there a discrete analogue? Yes
I will introduce it in a slightly more general context.

Polya's work



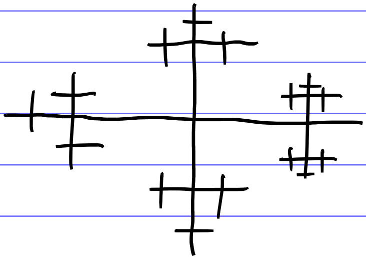
$$p^{(2n)}_{\mathbb{Z}}(0,0) = \frac{\binom{2n}{n}}{2^{2n}} \approx \frac{c_1}{\sqrt{n}}$$

paths starting at 0 and ending at 0, of length $2n$

$$p^{(2n)}_{\mathbb{Z}^2}(0,0) \approx \frac{c_2}{n}$$

Γ nilpotent of growth d. $p^{(2n)}_{\Gamma}(1_\Gamma, 1_\Gamma) \approx \frac{1}{n^{d/2}}$

On the other hand if we consider F_2



$$p^{(2n)}_{F_2}(1_{F_2}, 1_{F_2}) = \frac{4^n}{4^{2n}}$$

$$= \frac{1}{4^n} \approx \frac{1}{e^n}$$

Kesten (1950's) realized that

Γ is non-amenable $\iff$ $p_\Gamma^{2n} \approx \underbrace{e^{-n}}_{\text{smallest possible growth}}$

$$\sup_{x \in M} h_n(x,x) \approx P_{2n}(1,1)$$

as $t \to \infty$ as $n \to \infty$

Heat kernel RW on group

(Reversible) Markov Chain

X state space, (countable) set

$P = (p(x,y))_{x,y \in X}$ stochastic matrix, that is:

i) $0 \leq p(x,y) \leq 1$

ii) $\sum_{y \in X} p(x,y) = 1$.

P transition matrix (or transition operator)

$p(x,y) =$ "probability of moving from x to y"

a Markov chain is given by (X, P) and a choice of a starting point $x \in X$

$\Omega = X^{\mathbb{N}_{\geq 0}}$ $Z_n : \Omega \to X$ n-th projection

$$\mathbb{P}_x(Z_0 = x_0, Z_1 = x_1, \dots, Z_n = x_n)$$
$$= \delta_x(x_0)\, p(x_0, x_1) \dots p(x_{n-1}, x_n)$$

$\mathbb{P}_x$ a prob. on Ω.

$(Z_n) : \Omega \to X$ Markov chain

We will denote

$$p^{(n)}(x,y) = \mathbb{P}_x(Z_n = y)$$

= probability that starting at x we will arrive to y in n steps.

$= (P^n)_{x,y}$

(x,y)-entry of P^n.

Return probability (to the origin) will be

$$p^{(2n)}(x,x) \quad \text{or} \quad \sup_{x \in X} p^{(2n)}(x,x)$$

(X,P) is irred. if $\forall x,y \in X$, $\exists n$ s.t. $p^{(n)}(x,y) > 0$.

Reversible Markov Chains

(X,P) will be reversible if there exists m measure on X, $m: X \to (0,\infty)$ such that:

$$(\star) \quad m(x)p(x,y) = m(y)p(y,x) \quad \forall x,y \text{ in } X$$

$a(x,y) = m(x)p(x,y) = a(y,x)$ conductance

Why do we care about this condition?

$f \in \mathbb{R}^X$, $\quad P: \mathbb{R}^X \to \mathbb{R}^X$

$$Pf(x) = \sum_{y \in X} p(x,y) f(y)$$

$$P: \ell^2(X,m) \to \ell^2(X,m)$$

$(\star) \Rightarrow$ P is self-adjoint.

Self-adjoint op. are nice!

Examples:

- simple RW on a graph ($\overset{\text{random walk}}{\text{RW}}$)

  (V, E)

  $X = V$

  $$p(v,w) = \begin{cases} \dfrac{1}{\deg(v)} & \text{if } v \sim w \\ 0 & \text{if not} \end{cases}$$

  $m(v) = \deg(v)$

- from a MC, (X, P) we can also obtain a (weighted) graph

- RW on a group

  Γ countable group, $\mu \in \mathrm{Prob}(\Gamma)$

  μ symmetric, $\mu(\gamma) = \mu(\gamma^{-1}) \ \forall \gamma \in \Gamma$

  $$p(\gamma, \gamma s) = \mu(s)$$

  Sometimes you ask as well $\langle \operatorname{supp} \mu \rangle = \Gamma$

  $\Gamma = \langle S \rangle$, $S = S^{-1}$

Simple RW on $\mathrm{Cay}(\Gamma, S) =$ RW on Γ with

$$\mu = \frac{1_S}{|S|}.$$

$m(\gamma) = 1$.

<u>Obs:</u> In this case $p^{(n)}(e,e) = \mu^{(n)}(e)$

$m : \Gamma \times \Gamma \times \dots \times \Gamma \longrightarrow \Gamma \qquad m_*(\mu \otimes \dots \otimes \mu) = \mu^{(n)}$

(Γ, S_1), (Γ, S_2) $\langle S_1 \rangle = \langle S_2 \rangle = \Gamma \qquad p_{S_1}^{2n} \approx p_{S_2}^{2n}$?

Moreover, Γ (f.g. $\langle S\rangle = \Gamma$), $\nu, \mu \in \mathrm{Prob}(\Gamma)$ symmetric

s.t. $\sum_{\gamma\in\Gamma} |\gamma|_S^2\,\mu(\gamma),\ \sum_{\gamma\in\Gamma} |\gamma|_S^2\,\nu(\gamma) < \infty$,

then:

$$p_\mu^{2n} \approx p_\nu^{2n}$$

discrete case: Pittet, Saloff-Coste (2000)

Dretete $\to$ LC. case. vN-algebra

Does there exists a link between $p_\Gamma^{(2n)}$ and $J_{2,\Gamma}$? YES!

Coulhon–Grigoryan:

If $J_{2,\Gamma}$ satisfies condition (S), then we can recover p_Γ^{2n} from $J_{2,\Gamma}$

$$t = \int_1^{\gamma(t)} \frac{J_{2,\Gamma}(v)^2}{v}\,dv$$

cond. (S) = log. derivative of γ has at most pol. growth.

Then $p_\Gamma^{2n} \approx \frac{1}{\gamma(2n)}$

For all amenable gps we know, $J_{2,\Gamma}$ satisfies condition (S).

Q. I-invariance of $p_\Gamma^{(2n)}$ (w/o condition (S))

If $\varphi : \Gamma \to \Lambda$ a quasi-isometry, Γ, Λ f.g. gps. We then have that

$$p_\Gamma^{2n} \approx p_\Lambda^{2n}$$

proven by Saloff-Coste, Pittet 2000

Also true for transitive graphs!

$\Gamma = (V, E)$ is transitive if $\mathrm{Aut}(\Gamma) \curvearrowright V$ transitive

$$\inf_{x \in X} p_X^{2n}(x,x) \precsim \sup_{y \in Y} p_Y^{2n}(y,y)$$

<u>Q1</u>: if $\varphi : \Gamma \to \Lambda$, amenable gps
φ regular / coarse embedding implies
$p_\Lambda^{2n} \preceq p_\Gamma^{2n}$ (w/o condition (δ)) NOT KNOWN!

<u>Q2</u>: Q.I- invariance in the LC-setting
Also unknown!

<u>geometry</u> doesn't carry on RW's that easily!
<u>(QI-isometries)</u>

(X, P) reversible Markov chain $E(P) = \{(x,y), p(x,y) > 0\}$

$$m(x)\, p(x,y) = m(y)\, p(y,x) = a(x,y)$$

$$r[e] = \frac{1}{a(e^-, e^+)} \qquad e = \{e^-, e^+\} \in E(P)$$

resistance

$\ell^2(X, m)$, $\ell^2(E, r)$

$$\langle f, g \rangle = \sum_{x \in X} f(x)\, g(x)\, m(x)$$

$$\langle u, v \rangle = \sum_{e \in E} u(e)\, v(e)\, r(e)$$

$$\nabla : \ell^2(X, m) \longrightarrow \ell^2(E, r)$$

$$\nabla f(e) = \frac{f(e^+) - f(e^-)}{r(e)}$$

$$\nabla^*: \ell^2(E,r) \to \ell^2(X,m)$$

$$\nabla^* u(x) = \frac{1}{m(x)} \Big(\sum_{e: e^+ = x} u(e) - \sum_{e: e^- = x} u(e) \Big)$$

"loss" at x

$$\ell^2(X,m) \underset{\nabla^*}{\overset{\nabla}{\rightleftarrows}} \ell^2(E,r)$$

$$\Delta = \nabla^* \nabla : \ell^2(X,m) \to \ell^2(X,m)$$

$$\Delta = I - P \geq 0$$

$$\langle \Delta f, f \rangle = \langle \nabla^* \nabla f, f \rangle = \langle \nabla f, \nabla f \rangle \geq 0$$

$\|P\| \leq 1.$

$$D_p(f) = \langle \nabla f, \nabla f \rangle = \langle \Delta f, f \rangle$$

Dirichlet (quadratic) form.

$$D_p(f) = \frac{1}{2} \sum_{x,y \in X} |f(x) - f(y)|^2 m(x) p(x,y)$$

$p^{2n} \sim N(\lambda) = Tr_\Gamma(E_\lambda^\Delta)$, as $\lambda \to 0$

$\mathcal{L}\Gamma \subset B(\ell^2 \Gamma)$ Sauer, Bendikov, Pittet (2013)

We say that (X,P) satisfies IS_φ if $\exists$ $C > 0$ s.t.

$$\varphi(m(A)) \leq C\, a(\partial A)$$

$IS_d = IS_\varphi$, $\varphi(t) = t^{1-1/d}$

$D \subset E(P)$, $a(D) = \sum_{e \in D} a(e^-, e^+)$

PROP: $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ increasing, $J(t) = t/\varphi(t)$ increasing. If (X,P) satisfies IS_φ (with constant c) then (X,P) satisfies the following Nash ineq:

$$\|f\|_2^2 \leq \psi\left(\frac{\|f\|_1^2}{\|f\|_2^2}\right) \underbrace{\langle \nabla f, \nabla f \rangle}_{D_p(f)} \quad (NA_\psi)$$

for all $f \in C_0(X)$

$\psi(t) = 4c^2 J(4t)^2$

LEMMA:

$f \in \ell^2(X, m)$

i) $\|P^n f\|_2^2$ is decreasing

ii) $\|P^n f\|_2^2 - \|P^{n+1} f\|_2^2$ is also decreasing

Pf:

i) $\|P(P^n f)\|_2 \leq \|P^n f\|_2$ since $\|P\| \leq 1$.

ii) $\Delta_{P^2} = I - P^2 \geq 0$

$S = (I-P^2)^{1/2}$, $[S, P] = 0$ $\qquad P = P^*$

$$\begin{aligned} \|P^n f\|_2^2 - \|P^{n+1} f\|_2^2 &= \langle (I-P^2) P^n f, P^n f \rangle \\ &= \langle S^2 P^n f, P^n f \rangle \\ &= \langle P^n S f, P^n S f \rangle \\ &\leq \langle P^{n-1} S f, P^{n-1} S f \rangle \\ &= \|P^{n-1} f\|_2^2 - \|P^n f\|_2^2 . \end{aligned}$$

$$\|P^{2n}\|_{1\to\infty} = \sup_{x \in X} \frac{p^{(2n)}(x,x)}{m(x)} = \sup_{x,y \in X} \frac{p^{(2n)}(x,y)}{m(y)}$$

$\|$

$\|P^n\|_{1\to 2}^2$.